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HAMILTON ACTION AS A FUNCTION OF PHASE VARIABLES†

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The Hamilton action is obtained in explicit form as a function of the phase coordinates and time for certain classes of conservative systems. The application of the action function to problems of investigating the stability of conservative systems is considered. It is shown that from the representation of the Hamiltonian action function in explicit form one can draw useful conclusions regarding the qualitative nature of the behaviour of the solutions of the systems considered.

1. INTRODUCTION

As we know [1], Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}-\frac{\partial L}{\partial \mathbf{q}}=0 \tag{1.1}$$

can be obtained from the condition for the Hamilton action to be stationary

$$\delta S=\delta\int_0^{t_1} L(\tau,\mathbf{q},\dot{\mathbf{q}})d\tau=0 \tag{1.2}$$

which enables the action S to be regarded as a carrier of information on systems described by Eqs (1.1).

Starting from this fact and assuming below that system (1.1) is conservative, we replace the fixed value t_1 by the current value t in the expression for the action S, and we consider S as being a quantity which characterizes the true motion of the system—the action function

$$S=\delta\int_0^{t} L(\mathbf{q},\dot{\mathbf{q}})d\tau \tag{1.3}$$

We will confine ourselves to the case when $L(\mathbf{q},\dot{\mathbf{q}})\in C^2(D\times R^n)$ (D is a region in R_q^n) and the solutions of system (1.1) with origin at $D\times R^n$ are extended along the whole $t\in R$ axis. These conditions, without loss of generality in considering the problems investigated below, enable the action function S to be represented in the following form [2–4]

$$S=S^*(\tau,\mathbf{q}(\tau),\ \dot{\mathbf{q}}(\tau))\Big|_0^t\in C_{t\mathbf{q}\dot{\mathbf{q}}}^{(1,1,1)}(R\times D\times R^n) \tag{1.4}$$

If we use the Hamiltonian form of (1.1)

$$\dot{\mathbf{q}}=\partial H/\partial \mathbf{p},\quad \dot{\mathbf{p}}=\partial H/\partial \mathbf{p} \tag{1.5}$$

then, using (1.4) and (1.5), the action functions S can also be written in the form

$$S=S_1^*(\tau,\mathbf{q}(\tau),\ \mathbf{p}(\tau))\Big|_0^t\in C_{t\mathbf{q}\mathbf{p}}^{(1,1,1)}(R\times D\times R^n) \tag{1.6}$$

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The use of S in the form (1.4) or (1.6) turns out to be very effective when investigating the stability of conservative systems [2–4].

Since by the definition of the action function S itself

$$dS/dt = L(\mathbf{q},\dot{\mathbf{q}}) = L^*(\mathbf{q},\mathbf{p}) \tag{1.7}$$

by considering S in the form (1.6) as a function of the generalized coordinates $\mathbf{q}$, the momenta $\mathbf{p}$ and the time t, and using (1.5) and (1.7) we obtain the equation

$$\frac{\partial S}{\partial t} + \frac{\partial S}{\partial \mathbf{q}}\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial S}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{q}} = L^*(\mathbf{q},\mathbf{p}) \tag{1.8}$$

The latter can be interpreted as linear first-order partial differential equation which the function $S(t, \mathbf{q}, \mathbf{p})$ must satisfy.

As we know [5], every first-order partial differential equation has a solution which depends on an arbitrary function. Hence, the class of solutions of Eq. (1.8) is wider compared with the Hamilton action function S in the form (1.6). We do not rule out, in particular, that in a specific situation connected with the investigation of stability, it may turn out to be useful to use any solution of Eq. (1.8), which does not necessarily converge to the action function $S(t, \mathbf{q}, \mathbf{p})$. Nevertheless, we will confine ourselves below to considering classes of conservative systems for which it is the action function S that can be determined explicitly.

2. LINEAR SYSTEMS

We will assume that the initial Lagrangian L has the form

$$L = L_2 + L_1 + L_0 = \tfrac{1}{2}\dot{\mathbf{q}}^T A\dot{\mathbf{q}} + \mathbf{f}(\mathbf{q})\dot{\mathbf{q}} + L_0(\mathbf{q})$$

where A is a constant matrix, the quadratic form $\dot{\mathbf{q}}^T A\dot{\mathbf{p}}$ is positive-definite, the vector function $\mathbf{f}(\mathbf{q})$ is linear in $\mathbf{q}$, and $L_0(\mathbf{q})$ is a quadratic form.

By representing the Lagrangian L in the form

$$L = \mathbf{p}\dot{\mathbf{q}} - H = (\mathbf{pq})^{\cdot} - \mathbf{q}\dot{\mathbf{p}} - H = (\mathbf{pq})^{\cdot} + \mathbf{q}\frac{\partial H}{\partial \mathbf{q}} - H = (\mathbf{pq})^{\cdot} - L, \quad \mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} \tag{2.1}$$

where H is the Hamiltonian corresponding to system (1.1), by (1.3) and (2.1) we have

$$S = \tfrac{1}{2}\mathbf{pq}\big|_0^t \tag{2.2}$$

Thus, in the case of linear Lagrangian system one can calculate the Hamilton action function quite simply as a function of the phase variables and the time t. It is important to note here that to obtain expression (2.2) we do not use the fact that the system is conservative. This enables us to conclude that expression (2.2) also holds for non-autonomous Lagrangian linear systems.

Since the function S in the form (2.2) does not depend explicitly on t, from the expression

$$\frac{dS}{dt} = L = \frac{1}{2}\mathbf{p}^T A^{-1}\mathbf{p} + L_0 - \frac{1}{2}\mathbf{f}^T \mathbf{A}^{-1}\mathbf{f} \tag{2.3}$$

we arrive at a linear analogue of Pozharitskii's criterion [6], which was then generalized to extremely non-linear systems in [7, 8].

As we will show below, in the non-linear case also there are systems for which no problem arises in calculating the Hamilton action function S in explicit form.

3. NATURAL SYSTEMS

In the case considered the Lagrangian is defined by the expression [1]

$$L = T(\mathbf{q},\dot{\mathbf{q}}) - \Pi(\mathbf{q}) = \tfrac{1}{2}\dot{\mathbf{q}}^T A(\mathbf{q})\dot{\mathbf{q}} - \Pi(\mathbf{q}) \tag{3.1}$$

where the quantity $T(\mathbf{q}, \dot{\mathbf{q}})$ corresponds to the kinetic energy of the system and $\Pi(\mathbf{q})$ corresponds to the potential energy of the system.

We will assume that the following systems are satisfied: (1) A is a constant matrix, and (2) $\Pi(\mathbf{q}) = \Pi_k(\mathbf{q})$ is a homogeneous function of degree k: $\Pi_k(\lambda\mathbf{q}) = \lambda^k\Pi_k(\mathbf{q})$.

Since, taking (2.1) into account, in this case

$$L = (\mathbf{pq})^{\cdot} + \mathbf{q}\partial\Pi_k/\partial\mathbf{q} - H \tag{3.2}$$

from (3.2) and Euler's theorem on homogeneous functions, we have

$$L = (\mathbf{pq})^{\cdot} + k\Pi_k - H = (\mathbf{pq})^{\cdot} + k\Pi_k/2 + k(\Pi_k - h)/2 + kh/2 - H = (\mathbf{pq})^{\cdot} - kL/2 + h(k/2-1) \tag{3.3}$$

From (3.3) we obtain

$$S = \frac{2}{k+2}\mathbf{pq}\Big|_0^t + h\frac{k-2}{k+2}t, \quad k \neq -2. \tag{3.4}$$

As can be seen, the phase variables $\mathbf{q}$ and $\mathbf{p}$ and the time in expression (3.4) are separated. If, in particular, we put $k = 2$ in (3.4) we arrive at (2.2).

If $k = -2$, then in the scheme considered, it is not possible to determine the action function S in explicit form.

We can prove that the function S in the forms (2.2) and (3.4) provide examples of integrals of partial differential equation (2.8).

Since in the n-body problem [9] the Lagrangian L satisfies the above conditions 1 and 2, in this case, at least four non-singular trajectories, excluding collision between bodies, from (3.4) (in which we must put $k = -1$) we have

$$S = 2\mathbf{pq}\Big|_0^t - 3ht \tag{3.5}$$

In particular, we can conclude from (1.3) and (3.5) that Lagrange stability in the n-body problem can only occur when $T + \Pi = h < 0$, since along any trajectory of the system considered the Lagrangian L is non-negative and, therefore, when $h > 0$ and $t \to \infty$, $\mathbf{pq}$ also tends to infinity by (3.5).

This conclusion, which is based on representation of the Hamilton action function in explicit form, reflects the meaning of Jacobi's theorem on the negativeness of the energy for Lagrange-stable motions in the n-body problem [10, 11].

Expression (3.4) obtained above for the action function S relates to the whole region in which solutions of the natural system considered exist. If we confine ourselves to a local investigation of natural systems, for example, in the neighbourhood of the position of equilibrium, an expression for the action function can also be obtained with more general assumptions regarding $T(\mathbf{q}, \dot{\mathbf{q}})$ and $\Pi(\mathbf{q})$.

Thus, without loss of generality we will assume that the point $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ is the position of equilibrium in question. We will assume that $T(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^T A(\mathbf{q})\dot{\mathbf{q}}$, and the quadratic form $\dot{\mathbf{q}}^T A(\mathbf{0})\dot{\mathbf{q}}$ is positive definite. Suppose, moreover, that the function $\Pi(\mathbf{q})$ can be represented in the form

$$\Pi(\mathbf{q}) = \Pi_m(\mathbf{q}) + o(\Pi_m) \tag{3.6}$$

where Π_m is a homogeneous function of power $m > 0$.

Assertion 1. If the potential energy $\Pi(\mathbf{q})$ of the natural system at the point $\mathbf{q} = 0$ has a strict local maximum which is determined by the term Π_m in Eq. (3.6) then $\forall(\mathbf{q}, \dot{\mathbf{q}}) \in s_\varepsilon = \{(\mathbf{q}, \dot{\mathbf{q}}) \in D_{\mathbf{q}} \times R^n_{\dot{\mathbf{q}}}: \|\mathbf{q} \oplus \dot{\mathbf{q}}\| < \varepsilon\}$ and the Hamilton action function is given by the expression

$$S = \left[\frac{2}{m+2}\mathbf{pq}\Big|_0^t + h\frac{(m-2)}{m+2}t\right](1 + \eta(\varepsilon)) \tag{3.7}$$

$$\lim_{\varepsilon\to 0}\eta(\varepsilon)=0,\quad \mathbf{p}=\frac{\partial L}{\partial\dot{\mathbf{q}}}$$

Proof. Following the scheme described above we have

$$L=(\mathbf{pq})^{\cdot}-\frac{m}{2}L+\dot{\mathbf{q}}\partial T/\partial\dot{\mathbf{q}}+o(\|\mathbf{q}\|^m)+h\left(\frac{m}{2}-1\right)$$

Since $\dot{\mathbf{q}}\partial T/\partial\dot{\mathbf{q}}=o(T)$, the last equation can be represented in the form

$$L=(\mathbf{pq})^{\cdot}-\frac{m}{2}L+L[o(L)]+h\left(\frac{m}{2}-1\right) \tag{3.8}$$

$$o(L)=[o(\|\mathbf{q}\|^m)+o(T)]/L$$

Since under the conditions of Assertion 1 according to the theorem on the mean [12, p. 600] the following equality holds

$$\int_0^t L[o(L)]d\tau=\mu\int_0^t Ld\tau \tag{3.9}$$

where $\mu=o(L)|_{(\mathbf{q},\mathbf{p})\in s_\varepsilon}$, $t\in I\cap R^+$ and I is the maximum interval within which the solution $(\mathbf{q},\mathbf{p})$ belongs to the neighbourhood s_ε, then from (3.8) and (3.9) we arrive at (3.7).

Note. Lyapunov [13] used the function $V=\mathbf{pq}$ to prove instability under the conditions of Assertion 1. This function, apart from a constant factor, which, however, has no effect on the properties of the auxiliary function, corresponds to the primitive for the first term in (3.7). If we take into account the fact that the conclusion that the equilibrium in this case is unstable follows from the representation of the action function S itself in the form (3.7), since, on the one hand $L\geqslant|h|$ and on the other $|(m-2)/(m+2)|<1$ and, therefore, as $t\to\infty$ the quantity $\mathbf{pq}$, by (3.7) also tends to infinity, then the choice of $\mathbf{pq}$ as the Lyapunov function seems completely natural.

Assertion 2. If the potential energy $\Pi(\mathbf{q})$ is a uniform function of degree $m>0(\Pi(\mathbf{q})=\Pi_m(\mathbf{q}))$, then, when there is no local minimum of the function $\Pi(\mathbf{q})$ at the point $\mathbf{q}=0$ and $(\mathbf{q},\dot{\mathbf{q}})\in\Omega^-=\{(\mathbf{q},\dot{\mathbf{q}})\in s_\varepsilon: T(\mathbf{q},\dot{\mathbf{q}})+\Pi(\mathbf{q})=h<0\}$ we have equality (3.7).

To prove Assertion 2 it is sufficient to note that when $(\mathbf{q},\dot{\mathbf{q}})\in\Omega^-$ the Lagrangian L is positive, so that we can use the same discussion that was used to prove Assertion 1.

Under the conditions of Assertion 2, as in the previous case, the instability of the equilibrium follows from the representation of S in the form (3.7), since $L\geqslant|h|$ when $(\mathbf{q},\dot{\mathbf{q}})\in\Omega^-$, and hence $\mathbf{pq}$ approaches infinity, by (3.7), when $h<0$ and $t\to\infty$.

Thus, when the conditions of Assertions 1 and 2 are satisfied, which are identical with the conditions of Lyapunov's [13] and Chetayev's [14] theorems, respectively, on the instability of equilibrium, we can also obtain explicit expressions for the Hamilton action function S. However, they are of a local character and in a certain sense contain an element of uncertainty. This is due to the fact that, as regards the quantity $\eta(\varepsilon)$, we can only say that it is small and can be calculated in the neighbourhood of s_ε, but in practice we cannot specify exactly the point of the neighbourhood s_ε at which $\eta(\varepsilon)$ is satisfied.

4. THE STABILITY OF THE EQUILIBRIUM OF NATURAL SYSTEMS

The procedure for obtaining the action function S in the form (1.4) or (1.6) described in [2–4] assumed the property of system (1.1) to be the flow ([15], p. 347). When carrying out calculations for the function S of expressions (3.4) and (3.7) this property was in no way used. This enables us to conclude that for systems of this class, equalities (3.4) and (3.7), taking the method by which they were obtained into account, remains true when there are minimal limitations on the smoothness of the corresponding Lagrangians. In particular, they old when $L\in C^1$, when only the existence of solutions is ensured, and nevertheless the definition of the Hamilton action function retains its meaning.

We will confine ourselves below to a more detailed consideration of systems with a negative homogeneous potential having its own specific feature and has been less investigated compared with systems in which $\Pi(\mathbf{q})$ is a positive homogeneous function.

Theorem 1. Suppose $L(\mathbf{q}, \dot{\mathbf{q}}) \in C^1(D \times R^n)$ and the following conditions are satisfied: (1) $\Pi(\mathbf{q}) = \Pi_k(\mathbf{q})$, $\Pi_k(\lambda\mathbf{q}) = \lambda^k\Pi_k(\mathbf{q})$, (2) $k < 0$, and (3) $\partial T/\partial \mathbf{q} \equiv \mathbf{0}$.

Then, unstable positions of equilibrium of system (1.1), (3.1) correspond to critical points of the function $\Pi(\mathbf{q})$ in the region $D \subset R^n$.

Proof. By condition 1 we have

$$\mathbf{q}\partial\Pi/\partial\mathbf{q} = k\Pi$$

and, therefore, at critical points of the function $\Pi(\mathbf{q})$, if they exist, $\Pi(\mathbf{q}) = \mathbf{0}$.

Assuming further that the set ω of critical points $\Pi(\mathbf{q})$ is non-empty

$$\omega = \{\mathbf{q} \in D \subset R^n\colon \Pi(\mathbf{q}) = \mathbf{0},\ \partial\Pi/\partial\mathbf{q} = \mathbf{0}\} = \phi$$

and fixing one of them $\mathbf{q} = \mathbf{q}^*$, we can show that the position of equilibrium $\mathbf{q} = \mathbf{q}^*$, $\dot{\mathbf{q}} = \mathbf{0}$ under the conditions of Theorem 1 is unstable.

We note first that the relation $(\mathbf{pq})^{\cdot} = 2h$ follows from (3.3) if $k = -2$. Therefore, in the situation when $|h| \neq 0$, the expression pq can be regarded as a Lyapunov function, which enables us to conclude that the position of equilibrium $\mathbf{q} = \mathbf{q}$, $\dot{\mathbf{q}} = \mathbf{0}$ considered is unstable. Hence, we will distinguish two cases below: (1) $k < -2$ and (2) $-2 < k < 0$. Suppose $k < -2$. We will first assume that in the neighbourhood of the critical point $\mathbf{q} = \mathbf{q}^*$ we have $\Pi(\mathbf{q}) \geqslant 0$. Then, taking into account the fact that in this case $|L| \geqslant h > 0$, $(k-2)/(k+2) > 1$, using (3.4) we arrive at the conclusion that the quantity $\mathbf{pq}$ increases without limit as $t \to \infty$ and for $h > 0$ as small as desired. Consequently, the position of equilibrium $\mathbf{q} = \mathbf{q}^*$, $\dot{\mathbf{q}} = \mathbf{0}$ is unstable since the assumption contradicts the definition of stability.

If the potential energy $\Pi(\mathbf{q})$ can be taken to have negative values, then assuming $h < 0$ and noting that in this case $L \geqslant |h|$, $(k-2)/(k+2) > 1$, as above, using (3.4) we arrive at the conclusion that the quantity $|\mathbf{pq}|$ increase without limit as $t \to \infty$ and hence the equilibrium $\mathbf{q} = \mathbf{q}^*$, $\dot{\mathbf{q}} = \mathbf{0}$ is unstable.

When $-2 < k < 0$ we add the quantity ht to both sides of (3.4). We thereby obtain

$$\int\limits_0^t 2Td\tau = \frac{2}{k+2}\mathbf{pq}\Big|_0^t + \frac{2kh}{k+2}t \tag{4.1}$$

Assuming that $h > 0$ in (4.1) and noting that $k/(k+2) < 0$, we conclude from (4.1) that the quantity $\mathbf{pq}$ increases without limit when $t \to \infty$.

Thus, under the conditions of Theorem 1 the position of equilibrium $\mathbf{q} = \mathbf{q}^*$, $\dot{\mathbf{q}} = \mathbf{0}$ cannot be stable. Theorem 1 is proved.

Corollary 1. If we put $k = -1$ in condition 2 of Theorem 1, we arrive at the conclusion, which is the essence of Earnshaw's theorem [16], according to which a stable static configuration of electric charges is impossible.

Corollary 2. Critical points of a negatively homogeneous function of class C^1 cannot be points of its strict local extremum.

The particular feature of the n-body problem [9] and of the problem of the interaction of electric charges [16] is the fact that the corresponding potential energy $\Pi(\mathbf{q})$ satisfies Laplace's equation

$$\Delta\Pi(\mathbf{q}) = 0 \tag{4.2}$$

It turns out that if the constraint (4.2) holds, we can say that the equilibrium is unstable without the requirement for the potential to be homogeneous.

Theorem 2. Suppose $L(\mathbf{q}, \dot{\mathbf{q}}), \in C^2(D \times R^n)$ and $\Delta\Pi(\mathbf{q}) = 0$.

Then, isolated critical points of the function $\Pi(\mathbf{q})$ in the region $D \subset R^n_{\mathbf{q}}$, if they exist, are unstable positions of equilibrium of system (1.1), (3.1).

Proof. Assuming that the potential energy $\Pi(\mathbf{q})$ is defined apart from a constant, without loss of generality we can regard the critical point $\mathbf{q}^*$ considered as belonging to the zero-level set of the function $\Pi(\mathbf{q})$. We can conclude from the fact that $\Pi(\mathbf{q})$ is harmonic [17] that $\mathbf{q}^*$ cannot be a point of local extremum of the function $\Pi(\mathbf{q})$. We obtain as a consequence that

$$\omega = \{\mathbf{q} \in s_\varepsilon = \{\mathbf{q} \in R^n, \| \mathbf{q} - \mathbf{q}^* \| < \varepsilon\}: \Pi(\mathbf{q}) < 0\} \neq \phi$$

Since the critical point $\mathbf{q}^*$ of the function $\Pi(\mathbf{q})$ is assumed to be isolated, we can further use one of the theorems given in [2–4] on the instability of the equilibrium of conservative systems, and on the basis of these we can conclude that Theorem 2 holds.

Note. The theorem will remain true if the condition for the critical points of the function $\Pi(\mathbf{q})$ to be isolated is replaced by the condition that the level sets of the function $\Pi(\mathbf{q})$, containing critical points, are isolated, which results from the method of proving the theorems of instability, proposed in [2, 3], based on the use of Hamilton's action function.

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